EXAM 1 SOLUTIONS

WANT TO MAXIMIZE \( L(x, y) = xy \) SUBJECT
TO THE CONSTRAINT \( x^2 + y^2 = 1 \).

DEFINE MODIFIED FUNCTION,

\[
L^1(x, y, \lambda) = xy + \lambda(x^2 + y^2 - 1)
\]

\[
\frac{\partial L}{\partial x} = y + 2\lambda x = 0 \quad (1)
\]

\[
\frac{\partial L}{\partial y} = x + 2\lambda y = 0 \quad (2)
\]

\[
\frac{\partial L}{\partial \lambda} = x^2 + y^2 - 1 = 0 \quad (3)
\]

SOLVE (1) FOR \( y \):

\[
y = -2\lambda x
\]

AND SUBSTITUTE INTO (2)

\[
x + 2\lambda(-2\lambda x) = 0
\]

\[
\rightarrow x(1 - 4\lambda^2) = 0
\]

THE SOLUTION \( x = 0 \) IMPLIES \( y = 0 \),

WHICH DOES NOT SATISFY (3).

INSTEAD, WE USE

\[
\lambda = \pm \frac{1}{2}
\]

\[
\rightarrow y = -2(\pm \frac{1}{2}) x = \mp x
\]

\[
\rightarrow x^2 + (\mp x)^2 = 1 \rightarrow 2x^2 = 1
\]

\[
\rightarrow \left| x = \pm \frac{1}{\sqrt{2}}, y = \mp \frac{1}{\sqrt{2}} \right| \text{ (OPPOSITE SIGN CHOICES GIVE MINIMA)}
\]
2. If \( g(z) = g_0 - g'z \), then the equation of motion is (\( z \) positive upward):
\[
m \ddot{z} = -mg_0 + mg'z
\]
Multiply by \( \dot{z} \) and integrate over time:
\[
\int_0^t m \ddot{z} \dot{z} \, dt = -mg_0 \int_0^t \dot{z} \, dt + mg' \int_0^t z \dot{z} \, dt
\]
\[
\frac{1}{2} m \dot{z}^2 \bigg|_0^t = -mg_0 \dot{z} \bigg|_0^t + \frac{1}{2} mg' \dot{z}^2 \bigg|_0^t
\]
\[
\frac{1}{2} m \dot{z}^2 \bigg|_0^t = mg_0 \dot{z} \bigg|_0^t + \frac{1}{2} mg' \dot{z}^2 \bigg|_0^t
\]
\[
- \dot{V}_0^2 = -2g_0 h + g' h^2
\]
\[
\to g' h^2 - 2g_0 h + \dot{V}_0^2 = 0
\]
\[
h = \frac{g_0 \pm \sqrt{\left(\frac{g_0}{g'}\right)^2 - 4\frac{g'}{g_0} \dot{V}_0^2}}{2\frac{g'}{g_0}}
\]
\[
= \frac{g_0 \pm \sqrt{\left(\frac{g_0}{g'}\right)^2 - \frac{\dot{V}_0^2}{g'}}}{\frac{g'}{g_0}}
\]
\[
= \frac{g_0 \left[ 1 \pm \sqrt{1 - \frac{\dot{V}_0^2}{g' g_0^2}} \right]}{g'}
\]
In the limit \( g' \to 0 \),
\[
h \to \frac{g_0}{g'} \left[ 1 \pm \left(1 - \frac{1}{2} \frac{\dot{V}_0^2}{g_0^2} \right) \right]
\]
\[
\to \frac{g_0}{g'} \left[ 2 - \frac{1}{2} \frac{\dot{V}_0^2}{g_0^2} \right] \quad \text{OR} \quad \frac{\dot{V}_0^2}{2g_0} \quad \text{LIMIT}
CONCLUDE THAT MINUS ROOT IS PHYSICAL SOLUTION:

\[ h = \frac{g_0^2}{g_0^1} \left[ 1 - \sqrt{1 - \frac{V_0^2 g_0^1}{g_0^2}} \right] \]

b) A PHYSICAL SOLUTION REQUIRES

\[ 1 - \frac{V_0^2 g_0^1}{g_0^2} \geq 0 \]

\[ \Rightarrow g_0^1 \leq \frac{g_0^2}{V_0^2} \]

c) FROM NEWTON'S LAW OF GRAVITATION:
(Spherically Symmetric Earth)

\[ g(r) = \frac{GM_e}{r^2}, \quad r \geq R_e \]

\[ g_0 = g(R_e) = \frac{GM_e}{R_e^2} \]

TAYLOR EXPANSION:

\[ g(R_e + z) = g(R_e) + \left( \frac{dg}{dr} \right)_{r=R_e} z + \frac{1}{2} \left( \frac{d^2 g}{dr^2} \right)_{r=R_e} z^2 + \cdots \]

\[ = g_0 + \frac{2GM_e}{R_e^3} z \]

\[ \Rightarrow g_0^1 = \frac{2GM_e}{R_e^3} = \sqrt{\frac{2g_0}{R_e}} \]
(2) DISPLACE THE ROD FROM EQUILIBRIUM BY A SMALL ANGLE $\theta$:

\[ U_{\text{grav}} = mgR_{\text{cm}} = mg \frac{R}{2} \cos(\theta) \]
\[ \approx mg \frac{R}{2} (1 - \frac{\theta^2}{2}) \quad \theta \ll 1 \]

THE SPRING'S POTENTIAL ENERGY IS

\[ U_{\text{spring}} = \frac{1}{2} k x^2 = \frac{1}{2} k \theta^2 \]
\[ \approx \frac{1}{2} k R^2 \theta^2 \]

TOTAL POTENTIAL ENERGY IS

\[ U(\theta) = \frac{mg R}{2} + \left( \frac{1}{2} k R^2 - \frac{1}{4} mg R \right) \theta^2 \]

\[ \frac{dU}{d\theta} \bigg|_{\theta=0} = 0 \]

\[ \frac{d^2U}{d\theta^2} \bigg|_{\theta=0} = 2 \left( \frac{1}{2} k R^2 - \frac{1}{4} mg R \right) \]

STABLE EQUILIBRIUM REQUIRES \[ \frac{d^2U}{d\theta^2} \bigg|_{\theta=0} > 0 \]

\[ \Rightarrow \frac{1}{2} k R^2 \frac{1}{4} mg R > 0 \]
OR \[ \kappa > \frac{1}{2} \frac{mg}{L} \]

b) NOW ASSUME \[ \kappa = \frac{2mg}{L} \]

DEFINE BOTTOM OF ROD AS ORIGIN.

TORQUE ACTING ON ROD FROM SPRING IS

\[ \tau_{spring} = -kxL = -kL^2 \theta \]

GRAVITATION EXERTS A CW TORQUE:

\[ \tau_{grav} = mg (\frac{L}{2}) \sin(\theta) = \frac{mgL}{2} \theta \]

MOMENT OF INERTIA ABOUT PIVOT:

\[ I = \frac{1}{2} mL^2 \]

EQN. OF MOTION:

\[ I \ddot{\theta} = (-kL^2 + \frac{mgL}{2}) \theta \]

\[ = \left( -\frac{B}{2} mgL \right) \theta \]

\[ \rightarrow \ddot{\theta} = -\omega_0^2 \theta \]

WITH \[ \omega_0^2 = \frac{3}{2} \frac{mgL}{mL^2} \]

\[ \rightarrow \omega_0 = \frac{3}{\sqrt{2} L} \]